RESPONSE OF A NON-LINEAR DEVICE TO NOISE

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Abstract

A non-linear device R is connected in series with an admittance $Y(\omega)$ and a random noise voltage $v(t)$ (e.g., arising from thermal agitation) impressed across the combination. Assume that one knows (a) the current-voltage function of R, and (b) the admittance $Y(\omega)$. Then we can obtain approximate statistical information about the voltage $v_1(t)$ across the non-linear device: explicit formulas depending only on (a) and (b) can be given for the moments of all orders of $v_1(t)$, and similarly for its frequency spectrum.

1. A non-linear device R is connected in series with an admittance $Y(\omega)$ and a random noise voltage $v(t)$ (e.g., thermal agitation) impressed across the combination.

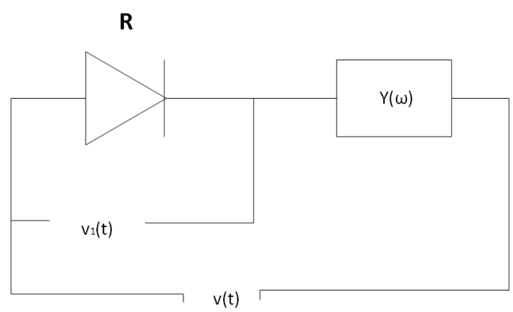

Figure 1.

Assume that one knows: (a) the current-voltage function of R and (b) the admittance of $Y(\omega)$, Then one can obtain approximate statistical information about the voltage $v_1(t)$ across the non-linear device: explicit formulas depending only on (a) and (b) can be given for the moments of all orders of $v_1(t)$, and similarly for its frequency spectrum.

2. There are two ideas involved. The first is to express $v_1(t)$ in terms of $v(t)$, assuming (a) and (b) known. The second idea is then to make use of the random nature of $v(t)$ to get the statistical information about $v_1(t)$.

The first idea employs an operator-series expansion for $v_1(t)$, and the second employs known averaging processes on Brownian functions.

3. In order to make matters definite we shall assume throughout the sequel that if the voltage across R is $v_1(t)$ the current through it is $v_1(t) + \bullet(v_1(t))^2$.

4. Let us denote by $A'(t)$ the indicial admittance corresponding to the frequency admittance $Y(\omega)$. Then it is well known that the current through $Y(\omega)$ is

$$\int_{-\infty}^{\infty} A'(t-\tau)(v(\tau) - v_1(\tau))\mathrm{d}\tau$$

Since current through R = current through $Y(\omega)$,

$$v_1(t) + \bullet(v_1(t))^2 = \int_{-\infty}^{\infty} A'(t-\tau)(v(\tau) - v_1(\tau))d\tau$$

Collecting terms in $v_1(t)$:

$$v_1(t) + \bullet(v_1(t))^2 + \int_{-\infty}^{\infty} A'(t-\tau)v_1(\tau)d\tau = \int_{-\infty}^{\infty} A'(t-\tau)v(\tau)d\tau \tag{1}$$

5. How we some to the first basic step of the paper, that of solving for $v_1(t)$ in terms of $v(t)$. To do this we assume [1] that

$$v_1(t) = \int_{-\infty}^{\infty} Q_1(t-\tau)v(\tau)d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_2(t-\tau_1, t-\tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_3(t-\tau_1, t-\tau_2, t-\tau_3)v(\tau_1)v(\tau_2)v(\tau_3)d\tau_1 d\tau_2 d\tau_3 \quad (2)$$

it is clear that we shall have secured the desired solution for $v_1(t)$ as soon as we have found $Q_1, Q_2, Q_3, \ldots$

6. Now substitute expression (2) for $v_1(t)$ in Eq.(1). We shall then equate linear part with linear part, quadratic part with quadratic part, etc.

7. First for the first degree terms in Eq.(1), we have

$$\int_{-\infty}^{\infty} Q_1(t-\tau)v(\tau)d\tau + \int_{-\infty}^{\infty} A'(t-\sigma)d\sigma \int_{-\infty}^{\infty} Q_1(\sigma-\tau)v(\tau)d\tau = \int_{-\infty}^{\infty} A'(t-\tau)v(\tau)d\tau \quad (3)$$

It is clear that (3) will be satisfied if

$$Q_1(t-\tau) + \int_{-\infty}^{\infty} A'(t-\sigma)d\sigma Q_1(\sigma-\tau) = A'(t-\tau) \quad (4)$$

Suppose Q_1 is the Fourier Transform of q_1; then recalling that $Y(\omega)$ is the Fourier transform if $A'(t)$, i.e.,

$$Q_1(t) = \int_{-\infty}^{\infty} q_1(\omega)\mathrm{e}^{j\omega t}d\omega \quad (5)$$

$$A'(t) = \int_{-\infty}^{\infty} Y(\omega)\mathrm{e}^{j\omega t}d\omega \quad (6)$$

Eq. (4) leads to

$$q_1(\omega)(1 + 2\pi Y(\omega)) = Y(\omega)$$

Hence,

$$q_1(\omega) = \frac{Y(\omega)}{1 + 2\pi Y(\omega)} \quad (7)$$

thereby determining $Q_1(t)$;

8. Equating the second degree terms in Eq.(1) to each other we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_2(t-\tau_1, t-\tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2 + \bullet\left(\int_{-\infty}^{\infty} Q_1(t-\tau)v(\tau)d\tau\right)^2 + \int_{-\infty}^{\infty} A'(t-\tau)d\tau \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_2(\tau-\tau_1, \tau-\tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2 = 0 \quad (8)$$

Eq. (8) will hold if

$$Q_2(t,\tau) - \bullet Q_2(\tau_1, \tau_2) = \int_{-\infty}^{\infty} A'(t-\tau)Q_2(\tau+\tau_1, \tau+\tau_2)d\tau \quad (9)$$

Suppose

$$Q_2(\tau_1, \tau_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{j(\omega_1\tau_1 + \omega_2\tau_2)} q_2(\omega_1, \omega_2)d\omega_1 d\omega_2 \quad (10)$$

i.e. $Q_2(\tau_1, \tau_2)$ is the double Fourier transform of $q_2(\omega_1, \omega_2)$. Recalling Eqs. (5) and (6), Eq. (10) leads to

$$q_2(\omega_1, \omega_2) + \bullet q_1(\omega_1)q_1(\omega_2) + 2\pi Y(\omega_1 + \omega_2)q_2(\omega_1, \omega_2) = 0$$

[1] In general one will also include a constant term Q_0

Hence

$$q_2(\omega_1,\omega_2) = -\frac{\bullet q_1(\omega_1)q_2(\omega_2)}{2\pi Y(\omega 1+\omega_2)+1} = -\frac{\bullet Y(\omega_1)Y(\omega 2)}{(1+2\pi Y(\omega_1))(1+2\pi Y(\omega_2))(1+2\pi Y(\omega_1+\omega_2))} \tag{11}$$

Note that q_2 contains the factor $\bullet$, indicating the effect of the non-linearity of the device R.

9. In the same way one computes $q_3(\omega_1,\omega_2,\omega_3)$:

$$\begin{aligned} q_3(\omega_1,\omega_2,\omega_3) &= \frac{2\pi q_1(\omega_1)q_2(\omega 2+\omega_3)}{1+2\pi Y(\omega_1+\omega_2+\omega 3)} \\ &= \frac{2\pi^2 Y(\omega_1)}{1+2\pi Y(\omega_1)} * \frac{Y(\omega_2)Y(\omega_3)}{(1+2\pi Y(\omega_2))(1+2\pi Y(\omega_3)(1+2\pi Y(\omega_2+\omega_3)} * \frac{1}{1+2\pi Y(\omega_1+\omega_2+\omega_3)} \end{aligned} \tag{12}$$

where the q_n's are defined in analogy with Eqs. (5) and (10), as multiple Fourier transforms of the Q_n i.e.:

$$Q_n(\tau_1,\ldots,\tau_n) = \int\ldots\int \mathrm{e}^{j\sum_k^n \omega_k\tau_k} q_n(\omega_1,\ldots,\omega_n)d\omega_1\ldots d\omega_n \tag{13}$$

Note that q_3 contains $\bullet^2$. Similarly q_4 will contain $\bullet^3$, and so on. Thus taking higher powers of $\bullet$ into account is equivalent to going out farther in the series of q's; this is the characteristic feature of perturbation methods.

10. At this point we interpose a formula which will be needed soon, it expresses the average of Q_n, n even, in terms of an average of q_n:

$$\begin{aligned} &\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} Q_n(\tau_1,\tau_1,\ldots,\tau_{\frac{n}{2}},\tau_{\frac{n}{2}})d\tau_1\ldots d\tau_{\frac{n}{2}} \\ &\qquad = (2\pi)^{\frac{n}{2}}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} q_n(\omega_1,-\omega_1,\ldots,\omega_{\frac{n}{2}},\omega_{\frac{n}{2}})d\omega_1\ldots d\omega_{\frac{n}{2}} \end{aligned} \tag{14}$$

11. Eqs. (7), (10), and (12) tell us the first three terms of (2). In other words, we have accomplished (to three terms) our first basic task, that of expressing $v_1(t)$, the voltage across the non-linear device R in terms of the admittance $Y(\omega)$ and the voltage $v(t)$ across the entire circuit.

12. So far we have said nothing about $v(t)$, but we are now ready to make use of the fact that $v(t)$ is a random voltage. This will constitute the second step of the paper, and will be accomplished by taking ***averages*** of the random voltages in accordance with known formulas. In these formulas the average is taken with respect to the parameter α which in going from 0 to 1 runs through all Brownian motions. $x(t,\alpha)$ is a properly named Brownian motion, whose differential is a random voltage and $K(t_1,\ldots,t_n)$ is a symmetric function of n variables:

$$\int_0^1 d\alpha \underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{\text{n times}} K(t_1,\ldots,t_n)dx(t_1,\alpha)\ldots dx(t_n,\alpha) = 0 \tag{15}$$

if n is odd,

$$\begin{aligned} &\int_0^1 d\alpha \underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{\text{n times}} K(t_1,\ldots,t_n)dx(t_1,\alpha)\ldots dx(t_n,\alpha) \\ &\qquad = (n-1)(n-3)\ldots 1 \underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{\frac{n}{2} times} K(t_1,t_1,t_2,t_2,\ldots,t_{\frac{n}{2}},t_{\frac{n}{2}})dt_1\ldots dt_{\frac{n}{2}} \end{aligned} \tag{16}$$

if n is even.

13. Referring to Eq. (2), we inquire as to the average of $v_1(t)$. We apply Eqs. (15) and (16) to the Q's, then express the result in terms of the q's by (14), and finally (Eqs. (7), (11), and (12)) expressing the q's in terms of the admittance $Y(\omega)$ we find that the firs non-vanishing term of the average of $v_1(t)$ is

$$-2\pi\int_{-\infty}^{\infty}\frac{Y(\omega)Y(-\omega)}{(1+2\pi Y(\omega))(1+2\pi Y(-\omega))(1+2\pi Y(\omega))}d\omega \tag{17}$$

14. Similarly the average of $(v_1(t))^2$ is

$$
\begin{aligned}
2\pi \int_{-\infty}^{\infty} & q_1(\omega)q_1(-\omega)d\omega \\
&+ 4\pi^2 \left\{ 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_2(\omega_1,\omega_2)q_2(-\omega_1,-\omega_2)d\omega_1 d\omega_2 + \int_{-\infty}^{\infty} q_2(\omega,-\omega)d\omega \right\}^2 \\
&\qquad + 12\pi^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_1(\omega_1)q_3(\omega_1,\omega_2,-\omega_3)d\omega_1 d\omega_2 \\
&+ 3\pi^2 \left\{ 6\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_3(\omega_1,\omega_2,\omega_3)q_3(-\omega_1,-\omega_2,-\omega_3)d\omega_1 d\omega_2 d\omega_3 \right. \\
&\qquad \left. +9\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_3(\omega_1,\omega 2,-\omega_2)q_3(-\omega_1,\omega_3,-\omega_3)d\omega_1 d\omega_2 d\omega_3 \right\} + \dots
\end{aligned} \tag{18}
$$

15. In the same way the higher moments of $v_1(t)$ may be computed, so also ma the moments of $v(t) - v_1(t)$, the voltage across $Y(\omega)$, be computed.

16. An average of much importance is that of $v_1(t)v_1(t+\sigma)$; this average is called the auto-correlation coefficient, and itd Fourier transform gives the frequency distribution of the square of the voltage. The auto-correlation coefficient is the average of

$$
\begin{aligned}
&\left\{ \int_{-\infty}^{\infty} Q_1(t-\tau)v(\tau)d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_2(t-\tau_1, t-\tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2) + \dots \right\} \\
&\quad * \left\{ \int_{-\infty}^{\infty} Q_2(t+\sigma-\tau)v(\tau)d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_2(t+\sigma-\tau_1, t+\sigma-\tau_2)v(\tau_1)v(\tau_2)d\tau_1 d\tau_2 + \dots \right\}
\end{aligned} \tag{19}
$$

which is equal to

$$
\begin{aligned}
2\pi \int_{-\infty}^{\infty} q_1(\omega)q_1(-\omega)\mathrm{e}^{-j\omega\sigma}d\sigma + 4\pi^2 \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 q_2(\omega_1,-\omega_1)q_2(\omega_2,-\omega_2) \\
+ 8\pi^2 \int_{-\infty}^{\infty} d\omega 1 \int_{-\infty}^{\infty} d\omega_2 q_2(\omega_1,\omega_2)q_2(-\omega_1,-\omega_2)\mathrm{e}^{-j\sigma(\omega_1+\omega_2)} + \dots
\end{aligned} \tag{20}
$$

Since the second term does not contain σ, it is a constant, representing a DC component. The third term is equal to

$$
8\pi^2 \int_{-\infty}^{\infty} \mathrm{e}^{-j\sigma\omega} \int_{-\infty}^{\infty} q_2(\omega, \omega-\omega_1)q_2(-\omega_1, \omega_1-\omega)d\omega_1
$$

Thus to the frequency spectrum

$$
2\pi q_1(\omega)q_1(-\omega)
$$

present with no rectification (i.e. $\bullet = 0$), there has been added

$$
8\pi^2 \int_{-\infty}^{\infty} q_2(\omega_1, \omega-\omega_1)q_2(-\omega_1, \omega_1-\omega)d\omega_1
$$

17. ***Critique.*** The method above, of first solving for the voltage across part of the circuit in terms of the entire voltage, and then getting statistical averages, is clearly quite general. The particular application of this method given here has two weaknesses however:

(i) the current-voltage relation of R is over-simplified.

(ii) in every practical case a filter of finite band-width precedes the rectifier-admittance combination of Fig. 1.

The problem in which (i) and (ii) have been taken into account can, and should be, set up and solved.